COMMENTS

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Comment on "Energy balance for a dissipative system"

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In a recent paper [Phys. Rev. E **48**, 1547 (1993)] Li, Ford, and O'Connel argue that the power lost through dissipation by a system in equilibrium with a thermal reservoir is compensated by the action of a fluctuation force that the reservoir exerts on the system. Their analysis is quantum mechanical and their expression for the power supplied by the reservoir remains positive definite even at absolute zero. This Comment outlines an analysis of a harmonic oscillator coupled to a loss mechanism and shows how the quantum-mechanical formalism should be interpreted to avoid unphysical conclusions. An explicit distinction is made between thermal and zero-point fluctuations, and the physical significance of the latter is discussed.

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It is well known that a loss mechanism (LM) that produces dissipation in a given system will also exert a fluctuating force on that system. In a recent paper [1], Li, Ford, and O'Connel discuss the relationship between the power supplied by the fluctuation force and the power lost to the LM. This energy exchange is analyzed quantum mechanically, and the fluctuations considered refer to both thermal and quantum fluctuations. However, by not distinguishing between the two, the authors obtain results that are misleading. They begin with a phenomenological, quantum-mechanical, generalized Langevin equation for a particle of mass m in a potential V,

$$m\ddot{x}+\int_{-\infty}^{t}dt'\mu(t-t')\dot{x}(t')+\frac{dV(x)}{dx}=F(t)\ ,$$

where $\mu(t)$ is the dissipation function and $F(t)$ is the fluctuation force exerted by the LM. They then consider a harmonic oscillator with $V=\frac{1}{2}m\omega_0x^2$, in equilibrium with the LM, and calculate "the expectation value of the instantaneous power supplied by the fluctuation force, P_F." The result is

$$P_F=\frac{\hbar}{2\pi}\int_{-\infty}^{\infty}d\omega\,\omega^3|\alpha(\omega)|^2[\mathrm{Re}\tilde{\mu}(\omega)]^2\coth\left[\frac{\hbar\omega}{2kT}\right]\ ,$$

where

$$\alpha(\omega)=[m(\omega_0^2-\omega^2)-i\omega\tilde{\mu}(\omega)]^{-1}$$

and

$$\tilde{\mu}(\omega)=\int_0^{\infty}dt\,\mu(t)e^{i\omega t}\ ,\quad \mathrm{Im}\omega>0\ .$$

As the authors state, "it is immediately clear $P_F>0$ always." In particular, this inequality also holds for $T=0$. The authors therefore imply that the LM—or essentially a thermal reservoir—can supply power when its temperature is absolute zero. It is true that, according to quantum mechanics, there do exist fluctuations at $T=0$, zero-point fluctuations, but these fluctuations cannot do work. (Otherwise, a system in the ground state would have to give up energy.) The following discussion shows how the quantum-mechanical formalism describing the coupling of a harmonic oscillator to a LM should be interpreted in order to yield physically reasonable conclusions.

The quantum mechanics of a harmonic oscillator with dissipation was discussed in several early papers on dissipation in quantum mechanics [2–4]. The oscillator considered there is one that describes the behavior of an electromagnetic mode in a cavity, of interest in quantum optics, for which the ratio of period to relaxation time is much smaller than unity. The Hamiltonian of the oscillator is given by

$$H_{\mathrm{osc}}=\tfrac{1}{2}\hbar\omega(q^2+p^2)\ , \tag{1a}$$

where ω is the resonant frequency, and q and p are the dimensionless coordinate and momentum, respectively, with $[q,p]=i$. An alternative useful form is

$$H_{\mathrm{osc}}=\hbar\omega(a^{\dagger}a+\tfrac{1}{2})\ , \tag{1b}$$

where a and $a^{\dagger}$ are the annihilation and creation operators, respectively, with $a=2^{-1/2}(q+ip)$, $a^{\dagger}=2^{-1/2}(q-ip)$, and $[a,a^{\dagger}]=1$. The LM has the properties of a thermal reservoir, is not specialized to a specific model, and is weakly coupled to the oscillator. With approximations based on the general properties of the LM and the smallness of the ratio of period to relaxation time, Langevin equations are derived for the oscillator variables. Assuming that the coupling between the oscillator and the LM begins at $t=0$, and in the absence of a prescribed driving force, the solutions, for $t \gg \omega^{-1}$, are given by [2]

$$q(t)=q^{(o)}(t)e^{-(1/2)\beta t} + \int_0^t dt_1 F(t_1) e^{-(1/2)\beta(t-t_1)}\cos\omega(t-t_1) , \quad (2a)$$

$$p(t)=p^{(o)}(t)e^{-(1/2)\beta t} - \int_0^t dt_1 F(t_1) e^{-(1/2)\beta(t-t_1)}\sin\omega(t-t_1) . \quad (2b)$$

The variables $q^{(o)}(t)$ and $p^{(o)}(t)$ refer to the free oscillator and contain the initial values. The effects of the LM are determined by the decay constant β and the Gaussian fluctuation force (a quantum-mechanical operator) specified by [5]

$$\langle F(t) \rangle = 0 , \quad (3a)$$

$$\langle F(t_1)F(t_2) \rangle = \beta \left\{ \frac{i}{\omega}\delta'(t_1-t_2) + 2\delta(t_1-t_2)\left[\frac{1}{2}+\frac{1}{e^{\hbar\omega/kT}-1}\right]\right\} . \quad (3b)$$

A calculation of the expectation value of the power absorbed by the oscillator, using Eq. (1a), yields

$$\begin{aligned}\frac{d}{dt}\langle H_{\text{OSC}}\rangle &= \tfrac{1}{2}\hbar\omega\langle q\dot{q}+p\dot{p}\rangle + \text{c.c.} \\ &= \tfrac{1}{2}\hbar\omega\{-\beta\langle q^2+p^2\rangle + \int_0^t dt_1 \langle \{F(t_1),F(t)\}\rangle e^{-(1/2)\beta(t-t_1)}\cos\omega(t-t_1)\} , \\ &= \hbar\omega\left[-\tfrac{1}{2}\beta\langle q^2+p^2\rangle + \beta\left[\frac{1}{2}+\frac{1}{e^{\hbar\omega/kT}-1}\right]\right] , \end{aligned} \quad (4)$$

where the notation $\{A,B\}=AB+BA$ is used. It appears that the first term in the square brackets indicates the rate of energy loss due to dissipation, and the second term indicates the rate of energy gain due to the action of the fluctuation force. Thus one might conclude from the formalism that even when the LM is at zero temperature it transfers energy to the oscillator—the conclusion of Li, Ford, and O'Connel. Since a system in the ground state has no energy to transfer, this cannot be a correct conclusion.

Let us consider, now, the expression of Eq. (1b) instead of that of Eq. (1a) for the Hamiltonian of the oscillator. We obtain

$$\frac{d}{dt}\langle H_{\text{OSC}}\rangle = \hbar\omega\langle a^\dagger\dot{a}+\dot{a}^\dagger a\rangle . \quad (5)$$

Noting that

$$a(t)=a^{(o)}(t)e^{-(1/2)\beta t} + \frac{1}{\sqrt{2}}\int_0^t dt_1 F(t_1)e^{-[(1/2)\beta+i\omega](t-t_1)} ,$$

we get

$$\frac{d}{dt}\langle H_{\text{OSC}}\rangle = \hbar\omega[-\beta\langle a^\dagger a\rangle + 2^{-1/2}\langle Fa+a^\dagger F\rangle] . \quad (6)$$

It is clear that the second term in the square brackets is the power (in units of $\hbar\omega$) that the LM transmits to the oscillator. A calculation yields

$$\frac{d}{dt}\langle H_{\text{OSC}}\rangle = \hbar\omega\left[-\beta\langle a^\dagger a\rangle + \beta\frac{1}{e^{\hbar\omega/kT}-1}\right] . \quad (7)$$

According to this equation, the LM transfers energy to the oscillators only for $T>0$, as is to be expected on physical grounds. There is no inconsistency, of course, between this equation and Eq. (4). If we write $\frac{1}{2}(q^2+p^2)=a^\dagger a+\frac{1}{2}$, Eq. (4) reduces to Eq. (6).

Neither Eq. (4) nor Eq. (6) depend on the existence of a state of equilibrium. They apply no matter what the values of the oscillator energy or the LM temperature may be. Since the quantities $\langle a^\dagger a\rangle$ and $(e^{\hbar\omega/kT}-1)^{-1}$ are independent, it is obvious that the loss of oscillator energy above that of the zero point and the gain of energy due to thermal fluctuations are independent processes. However, this independence does not apply to the "loss" of zero-point oscillator energy and the "gain" due to zero-point fluctuations of the LM. These always cancel in the formalism and are not part of the real physical processes of energy transfer. The oscillator cannot, in reality, lose its zero-point energy, just as the LM cannot, in reality, provide energy at $T=0$. The following question therefore arises: what effect *do* the zero-point fluctuations of the LM have on the oscillator?

The effect turns out to be twofold. In part, it is purely formal. If the LM did not display zero-point fluctuations, there would be no cancellation of the formal loss of zero-point energy by the oscillator. Thus the self-consistency of quantum mechanics requires it. In part, the effect is also physical. This is best seen by examining the correlation function $\langle\{q(t_1),q(t_2)\}\rangle$ for both the undamped and damped oscillator. For the undamped oscillator in an energy state $|n\rangle$, we have

$$\langle\{q^{(o)}(t_1),q^{(o)}(t_2)\}\rangle = (2n+1)\cos\omega(t_1-t_2) .$$

If coupling to the LM begins at $t=0$, one obtains

$$\langle \{q(t_1),q(t_2)\} \rangle = (2n+1)e^{-(1/2)\beta(t_1+t_2)}\cos\omega(t_1-t_2) + 2[e^{-(1/2)\beta|t_1-t_2|} - e^{-(1/2)\beta(t_1+t_2)}] \times [\tfrac{1}{2} + (e^{\hbar\omega/kT}-1)^{-1}]\cos\omega(t_1-t_2) ,$$

where an approximation based on $\beta/\omega \ll 1$ has been used. For the steady state, one has

$$\langle \{q(t_1),q(t_2)\} \rangle = 2[\tfrac{1}{2} + (e^{\hbar\omega/kT}-1)^{-1}]e^{-(1/2)\beta|t_1-t_2|} \times \cos\omega(t_1-t_2) .$$

Although the zero-point energy of the undamped and damped oscillators is the same, it is seen that the zero-point motion is different. For the undamped oscillator it is a pure sinusoidal oscillation, and for the damped oscillator it is a noisy sinusoidal oscillation. Since the zero-point fluctuations of the LM may be regarded as noise, we can interpret this effect as a *modulation* of the zero-point motion of the oscillator, a modulation which does not require work. Such "effortless" modulation effects, which have been discussed previously [6], are associated with quantum noise.

[1] X. L. Li, G. W. Ford, and R. F. O'Connel, Phys. Rev. E **48**, 1547 (1993).

[2] I. R. Senitzky, Phys. Rev. **119**, 670 (1960); **124**, 642 (1961).

[3] M. Lax, Phys. Rev. **145**, 110 (1966).

[4] I. R. Senitzky, Phys. Rev. **155**, 1387 (1967).

[5] In Ref. [2], the first term in the curly brackets of Eq. (3) is given as $iP(t_2-t_1)^{-1}$. Its replacement by the present expression, which has certain advantages, was suggested by Lax (Ref. [3]), and was shown to fit into the original approximation scheme in Ref. [4].

[6] I. R. Senitzky, Phys. Rev. A **48**, 4629 (1993).